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Abstract. A group G is said to have dense solitary subgroups if each non-empty open interval in its subgroup lattice $L(G)$ contains a solitary subgroup. In this short note, we find all finite groups satisfying this property.

1. Introduction

Let $\mathcal{X}$ be a property pertaining to subgroups of a group. We say that a group G has *dense $\mathcal{X}$-subgroups* if for each pair (H,K) of subgroups of G such that $H<K$ and H is not maximal in K, there exists a $\mathcal{X}$-subgroup X of G such that $H<X<K$. Groups with dense $\mathcal{X}$-subgroups have been studied for many properties $\mathcal{X}$: being a normal subgroup [10], a pronormal subgroup [15], a normal-by-finite subgroup [6], a nearly normal subgroup [7] or a subnormal subgroup [4].

In our note, we will focus on the property of being a solitary subgroup. We recall that a subgroup H of a group G is called a *solitary subgroup* of G if G does not contain another isomorphic copy of H. By Theorem 25 of [9], the set $\mathrm{Sol}(G)$ of solitary subgroups of G forms a lattice with respect to set inclusion, which is called the *lattice of solitary subgroups* of G. This has been investigated in several recent paper, such as [1, 5, 13].

We also recall that a *ZM-group* is a finite non-abelian group with all Sylow subgroups cyclic. By [8], such a group is of type

$$\mathrm{ZM}(m,n,r)=\langle a,b \mid a^m=b^n=1,\ b^{-1}ab=a^r\rangle,$$

where the triple (m,n,r) satisfies the conditions

$$\gcd(m,n)=\gcd(m,r-1)=1 \text{ and } r^n\equiv 1 \pmod m.$$

It is clear that $|\mathrm{ZM}(m,n,r)|=mn$ and $Z(\mathrm{ZM}(m,n,r))=\langle b^d\rangle$, where d is the multiplicative order of r modulo m, i.e.,

$$d=o_m(r)=\min\{k\in\mathbb{N}^* \mid r^k\equiv 1 \pmod m\}.$$

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The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [3]. Set

$$L=\left\{(m_1,n_1,s)\in\mathbb{N}^3\,:\,m_1|m, n_1|n, s<m_1, m_1|s\frac{r^n-1}{r^{n_1}-1}\right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\mathrm{ZM}(m,n,r))$ of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1,n_1,s)\in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)}=\bigcup_{k=1}^{\frac{n}{n_1}}\alpha(n_1,s)^k\langle a^{m_1}\rangle=\langle a^{m_1},\alpha(n_1,s)\rangle,$$

where $\alpha(x,y)=b^xa^y$, for all $0\le x<n$ and $0\le y<m$. Remark that $|H_{(m_1,n_1,s)}|=\frac{mn}{m_1n_1}$, for any s satisfying $(m_1,n_1,s)\in L$.

Our main result is stated as follows.

Theorem 1.1. *A finite group G has dense solitary subgroups if and only if G is either cyclic or a ZM-group $\mathrm{ZM}(m,n,r)$, where m is prime, $d=o_m(r)$ is prime and $n=d^\alpha p^\beta$ with p prime, $p\neq d$, $\alpha\ge 1$ and $\beta\in\{0,1\}$.*

Note that the symmetric group $S_3=\mathrm{ZM}(3,2,2)=\mathrm{SmallGroup}(6,1)$ and $\mathrm{ZM}(13,6,2)=\mathrm{SmallGroup}(78,4)$ are examples of such groups with $d=2$, $\alpha=1$, $p=3$ and $\beta=0$ or $\beta=1$, respectively. Also, the dicyclic group $\mathrm{Dic}_3=\mathrm{ZM}(3,4,2)=\mathrm{SmallGroup}(12,1)$ and $\mathrm{ZM}(13,12,12)=\mathrm{SmallGroup}(156,4)$ are examples of such groups with $d=\alpha=2$, $p=3$ and $\beta=0$ or $\beta=1$, respectively.

For the proof of Theorem 1.1, we need the following well-known result (see e.g. (4.4) of [12], II).

Theorem 1.2. *A finite p-group has a unique subgroup of order p if and only if it is either cyclic or a generalized quaternion 2-group.*

By a *generalized quaternion 2-group* we will understand a group of order 2^n for some positive integer $n\ge 3$, defined by

$$Q_{2^n}=\langle a,b\mid a^{2^{n-2}}=b^2, a^{2^{n-1}}=1, b^{-1}ab=a^{-1}\rangle.$$

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [8, 12]. For subgroup lattice concepts we refer the reader to [11].

2. Proofs of the main results

We start with two auxiliary results. The first one shows that cyclic groups are the unique p-groups with dense solitary subgroups.

Lemma 2.1. *Let G be a finite p-group with dense solitary subgroups. Then G is cyclic.*

Proof. Let $|G| = p^n$, where $n \in \mathbb{N}^*$. If $n \geq 2$, then G contains a subgroup H of order p^2. By hypothesis, between 1 and H there is a solitary subgroup K. Then $|K| = p$ and K is the unique subgroup of order p of G. By Theorem 1.2, it follows that G is either cyclic or a generalized quaternion 2-group. Assume that $G \cong Q_{2^n}$ for some $n \geq 3$. Then $|Z(G)| = 2$ and G has a subgroup $Q \cong Q_8$. Since all subgroups of order 4 of G are cyclic, we infer that there is no solitary subgroup X such that $Z(G) < X < Q$, contradicting the hypothesis. Thus G is cyclic, as desired. □

The second one describes the lattice of solitary subgroups of a ZM-group.

Lemma 2.2. *We have*

$$\mathrm{Sol}(\mathrm{ZM}(m, n, r)) = \left\{H_{(m_1,n_1,0)} \in L(\mathrm{ZM}(m, n, r)) \, : \, m_1 | r^{n_1} - 1\right\}.$$

Proof. By [2], two subgroups of $\mathrm{ZM}(m, n, r)$ are conjugate if and only if they have the same order. This implies that two subgroups of $\mathrm{ZM}(m, n, r)$ are isomorphic if and only if they are conjugate, and so the solitary subgroups of $\mathrm{ZM}(m, n, r)$ coincide with the normal ones. The result now follows from [14, Theorem 1]. □

We are now able to prove our main result.

Proof of Theorem 1.1. Assume that G has dense solitary subgroups. Then all Sylow subgroups of G have this property, and so they are cyclic by Lemma 2.1. If G is not cyclic, then there is a triple of positive integers (m, n, r) such that $G = \mathrm{ZM}(m, n, r)$. Note that $d = o_m(r) \neq 1$. Also, we have

$$\mathrm{Sol}(G) = \left\{H_{(m_1,n_1,0)} \in L(\mathrm{ZM}(m, n, r)) \, : \, m_1 | r^{n_1} - 1\right\} \tag{1}$$

by Lemma 2.2.

Assume that m is not prime. Since G is supersolvable, its subgroup $H_{(m,1,0)} = \langle b \rangle$ of index m is not maximal. By hypothesis, there is $H_{(m_1,n_1,0)} \in \mathrm{Sol}(G)$ with

$$H_{(m,1,0)} < H_{(m_1,n_1,0)} < H_{(1,1,0)} = G.$$

Then $n_1 = 1$ and m_1 is a proper divisor of m, a contradiction. Thus m is prime and the equality (1) becomes

$$\mathrm{Sol}(G) = \left\{H_{(1,n_1,0)} \, : \, n_1 \,|\, n\right\} \cup \left\{H_{(m,n_1,0)} \, : \, d \,|\, n_1\right\}, \tag{2}$$

that is $\mathrm{Sol}(G)$ consists of all subgroups containing $H_{(1,n,0)} = \langle a \rangle$ and of all subgroups contained in $H_{(m,d,0)} = \langle b^d \rangle = Z(\mathrm{ZM}(m, n, r))$.

Since the interval $(\langle b^d \rangle, \langle b \rangle)$ of $L(G)$ contains no solitary subgroup, we infer that $\langle b^d \rangle$ is a maximal subgroup of $\langle b \rangle$, implying that d is prime.

If $d = n$, we are done. Suppose that $d \neq n$. Then either $\langle b^{\frac{n}{d}} \rangle$ is a maximal subgroup of $\langle b \rangle$ or the interval $(\langle b^{\frac{n}{d}} \rangle, \langle b \rangle)$ of $L(G)$ contains a solitary subgroup. In the first case, we get

$$n = d^2 \text{ or } n = dp, \text{ where } p \text{ is a prime with } p \neq d.$$

In the second case, let $H_{(m,n_1,0)} \in \mathrm{Sol}(G)$ such that

$$\langle b^{\frac{n}{d}} \rangle < H_{(m,n_1,0)} < \langle b \rangle.$$

Then

$$d \,|\, \frac{n}{n_1} \,|\, n.$$

Since $d|n_1$, it follows that the set $\{d' \,:\, d \,|\, d' \,|\, n\}$ must contain divisors of $\frac{n}{d}$, and therefore $d \,|\, \frac{n}{d}$, that is $d^2 \,|\, n$. Let $n = d^\alpha n'$ with $\alpha \geq 2$ and $d \nmid n'$.

If $n' = 1$, we are done. Suppose that $n' \neq 1$. Since there is no divisor d' of $\frac{n}{d}$ such that $d^\alpha \,|\, d' \,|\, n$, we infer that d^α must be a maximal divisor of n. Thus n' is prime.

The converse is obvious, completing the proof. □

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